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# Squeezed Number Eigenstate of XYZ Heisenberg Antiferromagnetics Under a Magnetic Field

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By using an algebraic diagonalization method, the XYZ Heisenberg antiferromagnetics under an external magnetic field is studied in the framework of spin-wave theory. The energy eigenstates are shown to be squeezed number states and the energy eigenvalues are obtained in some cases. Some quantum properties of the energy eigenstates, and the connection of the model with the two-mode coupled harmonic oscillators are also discussed.

**KEY WORDS:** Squeezed number state; XYZ antiferromagnetics; algebraic diagonalization.

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## 1. INTRODUCTION

The Heisenberg model (Anderson 1951; Heisenberg, 1928), describing spin lattices with short-range interactions, plays an indispensable role in the study of the magnetism of solids, and many other related physical fields, such as non-local spin systems, quantum dot, nuclear spin, and so on (Burkard, 1999; Imamoglu, 1999; Loss, 1988). According to the sign of interaction intensity $J$, this model can be classified as ferromagnetic ($J > 0$) type and antiferromagnetic ($J < 0$) type, the latter is much complicated than the former in the eigenstate structure. Based on the interaction intensity along different space directions, the Heisenberg model can be classified as XXX, XXZ, XYZ, where the former two cases can be regarded as the special case of the latter one. In theoretical study, a basic task is to do the digonalization of the Hamiltonian. However, due to the complicacy of the many body problem, getting the exact solution is very difficult, and till now only some special cases can be exactly solved, such as one dimension XXX antiferromagnetic

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chain using the Bethe-Ansatz method (Bethe, 1930). In practice, the spin-wave theory is widely employed as an acceptable approximation, and the "Néel states" is regarded to be the approximate ground states of the antiferromagnetic case. For XXZ antiferromagnetic Heisenberg model, spin-wave theory has been well studied in an analytical way which can be found in standard textbooks (Callaway, 1976). However, for XYZ case, as far as we know, there is no result parallel to XXZ case in literature. The reason lies in that, although it's not very difficult in principle to deal with the problem through the traditional Bogoliubov-valatin transformation (Bogoliubov, 1958; Valatin, 1958), the solution form is too complicated, to be intuitive in physics, and one have to rely on pure numerical analysis, leading to the lack of physical transparency.

As is well known, algebraic method is very important in studying the system with certain dynamical group (Artoni, 1991; Kim, 1988; Pan, 2001, 2004). In this paper, an algebraic diagonalization method will be employed to do the digonalization of XYZ antiferromagnetics under an external magnetic field in spin-wave framework. The energy eigenstates which turn out to be squeezed number states (Nieto, 1997; Kim, 1989; Yuen, 1976) and the corresponding eigenvalues in some cases will be obtained. The paper is organized as follows. Firstly, the algebraic diagonalization method is briefly introduced in Section 2. In Section 3, we study the system and give the result of eigenstates and eigenvalues in some cases. The relation between algebraic method and Bogoliubov method is also discussed. In Section 4, we discuss some statistical properties of the eigenstates. In Section 5, we investigate the relation of the system and the two-mode coupled harmonic oscillators. Finally, some concluding remarks are given.

## 2. AN ALGEBRAIC DIAGONALIZATION METHOD

The algebraic diagonalization method introduced here is applicable for a class of rather general systems whose Hamiltonian possesses dynamical semisimple Lie-algebra structure, i.e.

$$H = \sum_i \varepsilon_i H_i + \sum_\alpha (\lambda_\alpha E_\alpha + \lambda_\alpha^* E_{-\alpha}), \tag{1}$$

where $\{H_i,\ E_\alpha,\ E_{-\alpha}\}$ is the standard Cartan-Weyl basis of a semi-simple Lie algebra $g$ satisfying the standard commutation relations.

Based on the Lie algebra $g$, we introduce a unitary operator

$$W(\xi) = \exp\left[\sum_{\alpha>0} (\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha})\right], \tag{2}$$

in which $\xi_\alpha$ denote parameter functions to be determined. In fact, operator $W(\xi)$ is a generalized displacement operator in the coset space $G/H$ (Zhang, 1990).

Utilizing the Baker-Campbell-Hausdorff (B-C-H) formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots,$$

and taking advantages of the standard commutation relations of semisimple Lie algebra $g$, the following identities can be obtained

$$W(\xi)^{-1} G_j W(\xi) = \sum_i \gamma_{ij}(\xi) G_i, \tag{3}$$

where $G_j$ stand for the generators $H_i$ and $E_{\pm\alpha}$, $\xi$ denotes the set of $\{\xi_\alpha\}$, $\gamma_{ij}(\xi)$ are parameter functions. This is a central step in the overall treatment of the problem. Equation (3) can directly be rewritten into the following form

$$W(\xi)^{-1} H W(\xi) = \sum_i \eta_i(\xi, \varepsilon, \lambda) H_i + \sum_{\alpha>0} [(\mu_\alpha(\xi, \varepsilon, \lambda) E_\alpha + \mu_\alpha^*(\xi, \varepsilon, \lambda) E_{-\alpha}]. \tag{4}$$

The non-Cartan generators $E_{\pm\alpha}$ in the right-hand side are not diagonal and can be eliminated through setting

$$\mu_\alpha(\xi, \varepsilon, \lambda) = 0 \tag{5}$$

for all parameters $\alpha$ which give constraints to the parameter-functions $\{\xi_\alpha\}$ and the parameters $\varepsilon$, $\lambda$. Then we get

$$H W(\xi) \mid Ref >= W(\xi) \sum_i \eta_i(\xi, \varepsilon, \lambda) H_i \mid Ref > . \tag{6}$$

Here $\mid Ref >$ denote the reference states required to be the common eigenstates of all $H_i$. Therefore, $\sum_i \eta_i(\xi, \varepsilon, \lambda) H_i$ can be substituted by functions acting on the reference states, i.e.,

$$H W(\xi) \mid Ref >= \omega(\xi, \varepsilon, \lambda, Ref) W(\xi) \mid Ref > . \tag{7}$$

This is the eigen-equation of Hamiltonian Equation (1), thus the algebraic diagonalization have been accomplished. State $W(\xi) \mid Ref >$ and $\omega$ are the energy eigenstates and eigenvalues of corresponding system, respectively.

Now we briefly illuminate the completeness of the solution. For related fundamental introduction, please see reference (Zhang, 1990, pp. 877–879). In fact, the state $\mid Ref >$ and the operator $W(\xi)$ here correspond to the state $\mid \Lambda, \Lambda >$ (the common eigenstate of $H_i$) and the displacement operator $\Omega$ in that reference, respectively. So, eigenstates $W(\xi) \mid Ref >$ correspond to states $\mid \Lambda, \Omega >$ in that reference which play the role of complete basis of state space.

## 3. SQUEEZED NUMBER EIGENSTATES OF XYZ ANTIFERROMAGNETICS UNDER AN EXTERNAL MAGNETIC FIELD

The Hamiltonian of XYZ Heisenberg antiferromagnetics under an external magnetic field $\mathbf{B} = B\hat{e}_z$ (along $z$ axis) is described by

$$H = -J\sum_{\langle i,j\rangle}\left(\eta_x S_i^x S_j^x + \eta_y S_i^y S_j^y + S_i^z S_j^z\right) + \sum_i \mathbf{B}\cdot\mathbf{S}_i \qquad (J\langle 0, \quad \eta_x, \eta_y\rangle 0). \tag{8}$$

where the notation $\langle i, j\rangle$ denote the nearest neighbor bonds. We employ the traditional two-sublattice treatment, i.e., the spin directions are upwards for sites on sublattice A and downwards for sublattice B, and then apply Holstein-Primakoff transformation (Holstein, 1949):

$$S_a^z = -s + a^\dagger a, \qquad s_b^z = s - b^\dagger b,$$

$$S_a^\dagger = (2s)^{1/2}a^\dagger(1 - a^\dagger a/2s)^{1/2}, \quad S_b^\dagger = (2s)^{1/2}(1 - b^\dagger b/2s)^{1/2}b, \tag{9}$$

$$S_a^- = (S_a^\dagger)^\dagger, \qquad S_b^- = (S_b^\dagger)^\dagger,$$

where $a^\dagger$, $a$ ($b^\dagger$, $b$) can be regarded as the creation and annihilation operators of boson on sublattice A (sublattice B). The particle numbers $a^\dagger a$, $b^\dagger b$ can't exceed $2s$.

In low temperature and low excitation condition, $\langle a^\dagger a\rangle$, $\langle b^\dagger b\rangle \ll s$, so the non-linear interaction in Hamiltonian Equation (8) can be reasonable ignored (Kittel, 1963). Based on this "big s" approximation, transforming the operators into momentum space, and leaving out the biquadratic terms, we get

$$H = 2zsJ\left[Ns - \frac{1}{2}\left(\sum_{\mathbf{k}}\mathcal{H}_{\mathbf{k}} - 1\right)\right], \tag{10}$$

$$\begin{aligned}\mathcal{H}_{\mathbf{k}} = {} & a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{-\mathbf{k}}^\dagger b_{-\mathbf{k}} + \upsilon_{\mathbf{k}}(a_{\mathbf{k}}b_{-\mathbf{k}}^\dagger + a_{-\mathbf{k}}b_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger b_{-\mathbf{k}} + a_{-\mathbf{k}}^\dagger b_{\mathbf{k}}) \\ & + \rho_{\mathbf{k}}(a_{\mathbf{k}}b_{\mathbf{k}} + a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger + a_{-\mathbf{k}}b_{-\mathbf{k}} + a_{-\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger) + \mu(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \\ & + b_{-\mathbf{k}}^\dagger b_{-\mathbf{k}} - a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}),\end{aligned} \tag{11}$$

with

$$\upsilon_{\mathbf{k}} = \frac{\eta_x - \eta_y}{2}\gamma_{\mathbf{k}}, \qquad \rho_{\mathbf{k}} = \frac{\eta_x + \eta_y}{2}\gamma_{\mathbf{k}}, \qquad \gamma_{\mathbf{k}} = \frac{1}{z}\sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}}, \qquad \mu = \frac{B}{2zsJ}. \tag{12}$$

Here $\mathbf{R}$ is a vector connecting an atom with its nearest neighbor, and the sum runs over the $z$ nearest neighbors. $\mathbf{k}$ is restricted in the Brillouin zone. 2N is total number of the lattices.

$\mathcal{H}_{\mathbf{k}}$ can be expressed as the linear combination of six generators of Lie algebra $so(3, 2)$, i.e.,

$$\mathcal{H}_{\mathbf{k}} = E_3^{\mathbf{k}} + \mu F_3^{\mathbf{k}} + \rho_{\mathbf{k}}(E_+^{\mathbf{k}} + E_-^{\mathbf{k}}) + \upsilon_{\mathbf{k}}(F_+^{\mathbf{k}} + F_-^{\mathbf{k}}), \tag{13}$$

where the generators take the following forms

$$\begin{aligned} E_+^{\mathbf{k}} &= a_{\mathbf{k}}^+ b_{\mathbf{k}}^+ + a_{-\mathbf{k}}^+ b_{-\mathbf{k}}^+, & F_+^{\mathbf{k}} &= a_{\mathbf{k}} b_{-\mathbf{k}}^+ + a_{-\mathbf{k}} b_{\mathbf{k}}^+, \\ E_-^{\mathbf{k}} &= a_{\mathbf{k}} b_{\mathbf{k}} + a_{-\mathbf{k}} b_{-\mathbf{k}}, & F_-^{\mathbf{k}} &= a_{\mathbf{k}}^+ b_{-\mathbf{k}} + a_{-\mathbf{k}}^+ b_{\mathbf{k}}, \\ E_3^{\mathbf{k}} &= \frac{1}{2}(n_{\mathbf{k}}^a + n_{-\mathbf{k}}^a + n_{\mathbf{k}}^b + n_{-\mathbf{k}}^b + 2), & F_3^{\mathbf{k}} &= \frac{1}{2}(n_{\mathbf{k}}^b + n_{-\mathbf{k}}^b - n_{\mathbf{k}}^a - n_{-\mathbf{k}}^a). \end{aligned} \tag{14}$$

The other four generators of $so(3, 2)$ are

$$U_+^{\mathbf{k}} = b_{-\mathbf{k}}^+ b_{\mathbf{k}}^+, \qquad U_-^{\mathbf{k}} = b_{\mathbf{k}} b_{-\mathbf{k}}, \qquad V_+^{\mathbf{k}} = a_{-\mathbf{k}}^+ a_{\mathbf{k}}^+, \qquad V_-^{\mathbf{k}} = a_{\mathbf{k}} a_{-\mathbf{k}}. \tag{15}$$

The non-vanishing commutation relations of this algebra read

$$\begin{aligned} &[E_+, E_-] = -E_3, \quad [E_3, E_\pm] = \pm E_\pm, \\ &[F_+, F_-] = F_3, \quad [F_3, F_\pm] = \pm F_\pm, \\ &[E_3, U_\pm] = \pm U_\pm, \quad [F_3, U_\pm] = \mp U_\pm, \\ &[E_3, V_\pm] = \pm V_\pm, \quad [F_3, \quad V_\pm] = \pm V_\pm, \\ &[E_\pm, V_\mp] = \mp F_\mp, \quad [F_\pm, U_\pm] = \pm E_\pm, \\ &[E_\pm, F_\pm] = \mp V_\pm, \quad [V_+, V_-] = -(E_3 + F_3), \\ &[F_\pm, V_\mp] = \mp E_\mp, \quad [E_\pm, U_\mp] = \mp F_\pm, \\ &[E_\pm, F_\mp] = \mp U_\pm, \quad [U_+, U_-] = -(E_3 - F_3). \end{aligned}$$

One can see that $\{E_+^{\mathbf{k}}, E_-^{\mathbf{k}}, E_z^{\mathbf{k}}\}$ forms an $so(2, 1) \approx su(1, 1)$ subalgebra, while $\{F_+^{\mathbf{k}}, F_-^{\mathbf{k}}, F_z^{\mathbf{k}}\}$ forms an $so(3) \approx su(2)$ one. When $\eta_x = \eta_y$, the system will reduce to the $su(1, 1)$ case (Xie, 2002).

Hamiltonian Equation (13) possesses the form of Equation (1), then one can do the diagonalization of Hamiltonian Equation (13) through the algebraic diagonalization method introduced in Section 2. Note that the reference state $| Ref >$ is required to be the common eigenstate of $E_3^k$ and $F_3^k$, $| Ref >$ should take the form of number state $| n_a^k, n_b^k >$. So, the eigenstates $W(\xi) | Ref >$ are revealed to be squeezed number state, which had been studied in quantum optics field (Kim, 1989; Nieto, 1997; Yuen, 1976).

In the calculation process of the diagonalization, we found that, if a general full-parameter $W(\xi)$ is set, the infinite progressions induced by the B-C-H formula can't replaced by analytical functions. This lead to huge difficult even for number

calculation. For the purpose of studying the formal analytical solutions and the general property of the squeezed number eigenstate, we set

$$W(r,\theta) = \exp\{r[\cos\theta E_+ + \sin\theta(V_+ - U_+) - h.c.]\}. \tag{16}$$

Utilizing the commutation relations of $so(3,2)$ and the B-C-H formula, after a lengthy calculation, we get

$$W^\dagger(r,\theta)\mathcal{H}_{\mathbf{k}}W(r,\theta) = \omega_E^{\mathbf{k}}E_3 + \omega_F^{\mathbf{k}}F_3, \tag{17}$$

where

$$\omega_E^{\mathbf{k}} = \cosh 2r + \rho_{\mathbf{k}}\cos\theta\sinh 2r, \tag{18}$$

$$\omega_F^{\mathbf{k}} = \mu(1 + 2\sin^2\theta\sinh^2 r) + \upsilon_{\mathbf{k}}\sin 2\theta\sinh^2 r, \tag{19}$$

with the constraint equations

$$\cos\theta\sinh 2r + \rho_{\mathbf{k}}(\cosh^2 r - \cos 2\theta\sinh^2 r) = 0, \tag{20}$$

$$\mu\sin 2\theta\sinh^2 r + \upsilon_{\mathbf{k}}(\cosh^2 r + \cos^2\theta\sinh^2 r) = 0, \tag{21}$$

$$\mu\sin\theta\sinh 2r + \rho_{\mathbf{k}}\sin 2\theta\sinh^2 r - \upsilon_{\mathbf{k}}\cos\theta\sinh 2r = 0. \tag{22}$$

The role of these equations is to eliminate the non-Cartan generators appearing originally on the right-side of Equation (17). Note that there are 5 parameters in Equations (20)–(22), so one of the parameters $\rho_{\mathbf{k}}$, $\upsilon_{\mathbf{k}}$, $\mu$ should be not independent. Acting both sides of Equation (17) on reference number state $|n_a^{\mathbf{k}}, n_b^{\mathbf{k}}\rangle$, we get

$$\mathcal{H}_{\mathbf{k}}|R_{\mathbf{k}}>= \left(\omega_a^{\mathbf{k}}n_a^{\mathbf{k}} + \omega_b^{\mathbf{k}}n_b^{\mathbf{k}} + \frac{1}{2}\omega_E^{\mathbf{k}}\right)|R_{\mathbf{k}}>, \tag{23}$$

where

$$|R_{\mathbf{k}}>= W(r_{\mathbf{k}},\theta_{\mathbf{k}})|n_a^{\mathbf{k}}, n_b^{\mathbf{k}}\rangle, \tag{24}$$

$$\omega_a^{\mathbf{k}} = \frac{\omega_E^{\mathbf{k}} + \omega_F^{\mathbf{k}}}{2}, \qquad \omega_b^{\mathbf{k}} = \frac{\omega_E^{\mathbf{k}} - \omega_F^{\mathbf{k}}}{2}. \tag{25}$$

Equation (23) is the eigen-equation. The squeezed number state $W(r_{\mathbf{k}},\theta_{\mathbf{k}})|n_a^{\mathbf{k}}, n_b^{\mathbf{k}}\rangle$ is the eigenstate of the system, and $\omega_a^k$ and $\omega_b^k$ are the energies of the two different magnons respectively. It is shown that the magnetic field lift the magnon degeneracy. In the study of the number calculation, we find usually their exist several solutions, but only one solution give real energy eigenvalue. For instance, in a special case the reasonable solution is, $\rho = 0.1$, $\upsilon = -0.0006$, $\mu = 0.0383$, $\theta = 1.2506$, $r = -0.1636$, $\omega_a = 0.5418$, $\omega_b = 0.5017$.

Now we discuss the relation between the algebraic diagonalization method used here and the traditional Bogoliubov-Valatin transformation. To a certain

extend, these two methods are equivalent. The Bogoliubov-Valatin unitary transformation is known to acts on single creation or annihilation operator. In our case, the Lie algebra generators are composed of the quadratic form of the creation and annihilation operators. So, the unitary transformation acted on the Lie algebra generators corresponds to the product of two unitary transformations acting on single creation or annihilation operator respectively. This fact connects the two methods, and leads to that the eigenvalues should be the same and the energy eigenstates can be connected through considerable transformation. In fact, the unitary operator $W(r, \theta)$ (Equation (16)) provides a two-mode squeezing transformation,

$$\alpha_k = W^{-1}(r,\theta)a_k W(r,\theta) = a_k \cosh r + (\cos\theta b_k^\dagger - \sin\theta a_k^\dagger)\sinh r, \quad (26)$$

$$\beta_k = W^{-1}(r,\theta)b_k W(r,\theta) = b_k \cosh r + (\cos\theta a_k^\dagger + \sin\theta b_k^\dagger)\sinh r. \quad (27)$$

Here $\alpha_k$ and $\beta_k$ are new bosonic operators. In fact, the anti-transformation of Equations (26) and (27) can just be regarded as a corresponding Bogoliubov transformation. For briefness, we would not write them out here. The squeezed number form is $(\omega_a^{\mathbf{k}} n_a^{\mathbf{k}} + \omega_b^{\mathbf{k}} n_b^{\mathbf{k}} + \frac{1}{2}\omega_E^{\mathbf{k}})W(r_{\mathbf{k}}, \theta_{\mathbf{k}})|n_a^{\mathbf{k}}, n_b^{\mathbf{k}}\rangle$, where the $n_a^{\mathbf{k}}, n_b^{\mathbf{k}}$ are the eigenvalues of the particle number operators of $a_k^\dagger a_k, b_k^\dagger b_k$. The corresponding Bogoliubov form is $[\omega_\alpha^{\mathbf{k}}(\alpha_k^\dagger\alpha_k + \frac{1}{2}) + \omega_\beta^{\mathbf{k}}(\beta_k^\dagger\beta_k + \frac{1}{2})]|0, 0\rangle$, where the $\alpha_k$ and $\beta_k$ is the bosonic operators of the quasi-particles. It's apparent that the two methods work in different operator space, one is for "quasi-particle," the other is for "real particle."

## 4. SOME STATISTICAL PROPERTIES OF EIGENSTATE

In this section, we discuss some statistical properties of the eigenstate as squeezed number state. The average magnetization of sublattice A and B take the forms

$$\langle s_B^z \rangle = Ns - \sum_{\mathbf{k}} \langle b^\dagger b \rangle_{\mathbf{k}}, \quad (28)$$

$$\langle s_A^z \rangle = -Ns + \sum_{\mathbf{k}} \langle a^\dagger a \rangle_{\mathbf{k}}, \quad (29)$$

where,

$$\langle b^\dagger b \rangle_{\mathbf{k}} = \frac{1}{2}\cosh 2r\left(n_a^{\mathbf{k}} + n_a^{-\mathbf{k}} + n_b^{\mathbf{k}} + n_b^{-\mathbf{k}} + 2\right) + \frac{1}{2}(1 + 2\sin^2\theta\sinh^2 r) \times \left(n_b^{\mathbf{k}} + n_b^{-\mathbf{k}} - n_a^{\mathbf{k}} - n_a^{-\mathbf{k}}\right) - 1,$$

$$\langle a^\dagger a \rangle_{\mathbf{k}} = \frac{1}{2}\cosh 2r\left(n_a^{\mathbf{k}} + n_a^{-\mathbf{k}} + n_b^{\mathbf{k}} + n_b^{-\mathbf{k}} + 2\right) + \frac{1}{2}(1 + 2\sin^2\theta\sinh^2 r) \times \left(n_a^{\mathbf{k}} + n_a^{-\mathbf{k}} - n_b^{\mathbf{k}} - n_b^{-\mathbf{k}}\right) - 1.$$

When the particle numbers $n_a^{\mathbf{k}}$ and $n_b^{\mathbf{k}}$ vanish, the eigenstate becomes squeezed vacuum state. In this state,

$$\langle b^\dagger b\rangle_{\mathbf{k}} = \langle a^\dagger a\rangle_{\mathbf{k}} = \cosh 2r - 1. \tag{30}$$

One can see, even in the squeezed vacuum state, i.e., the ground state, the spin direction of different sites on a sub-lattice is not necessarily the same; this property of disordering has relations with the zero-point motion of the magnon in fact. Moreover, in the squeezed vacuum state, the influence of the external magnetic field upon the agree of average spin reversal of the two sub-lattice is inviolable the same; while in other squeezed number states, this property becomes true only in the condition that $n_a^{\mathbf{k}} + n_a^{-\mathbf{k}} = n_b^{\mathbf{k}} + n_b^{-\mathbf{k}}$.

Now we study the other quantum effects from the point view of quantum optics. For simplicity, hereafter only squeezed vacuum state is mentioned. The second-order correlation functions of the two excitation modes can be readily obtained as

$$g_b^{\mathbf{k}(2)} = g_a^{\mathbf{k}(2)} = \frac{\langle a_{\mathbf{k}}^{\dagger 2} a_{\mathbf{k}}^2\rangle}{\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle^2} = 2 + \sin\theta \coth^2 r, \tag{31}$$

$$g_{ab}^{\mathbf{k}\ (2)} = \frac{\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\rangle}{\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\rangle} = 1 + 2\cos\theta \coth^2 r. \tag{32}$$

The Mandel $Q$ parameters (Mandel, 1979, 1986) read

$$Q_b^{\mathbf{k}} = Q_a^{\mathbf{k}} = \frac{\langle(\Delta a_{\mathbf{k}}^\dagger a_{\mathbf{k}})^2\rangle}{\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle} - 1 = \sinh^2 r + \sin\theta \cosh^2 r. \tag{33}$$

Equation (33) illustrates that in squeezed vacuum state, each of the three statistics, sub-Poissonian ($Q > 0$), Poissonian ($Q = 0$), super-Poissonian ($-1 \leq Q < 0$, nonclassical state) are all possible to exist in the two magnon modes, relying on the values of the parameters $r$ and $\theta$, which are determined by the parameters $\upsilon_{\mathbf{k}}$, $\rho_{\mathbf{k}}$, $\mu$.

If the parameter $I^{\mathbf{k}} = \sqrt{g_b^{\mathbf{k}(2)} g_a^{\mathbf{k}(2)}}/g_{ab}^{\mathbf{k}\ (2)} - 1 < 0$, then the Cauchy-Schwartz inequality (CSI) (Agarwal, 1988) is violated, and the correlation between the two $\mathbf{k}$-magnon modes is nonclassical. In squeezed vacuum state,

$$I^{\mathbf{k}} = \frac{1 + (\sin\theta - 2\cos\theta)\coth^2 r}{1 + 2\cos\theta\ \coth^2 r}, \tag{34}$$

it can be found that larger squeezing parameter $r$ leads to smaller possibility of the violation of CIS, i.e. the achievement of nonclassical correlation. This is not surprising since larger squeeze parameter $r$ corresponds to larger particle number, tending to approaching classical case.

## 5. CONNECTION WITH THE TWO-DIMENSION COUPLED HARMONIC OSCILLATORS

For two different physical systems, if their Hamiltonian can be written into the same combination of the generators of a Lie algebra completely through different realizations of this Lie algebra, it's reasonable to believe that the two systems possess the same eigenvalue structure. Now we consider the relation of XYZ antiferromagnetic Heisenberg model under an external magnetic field and two-dimension coupled harmonic oscillators. Since the $so(3, 2)$ algebra can also be constructed in the form

$$\begin{aligned} &E_+ = a_1^\dagger a_2^\dagger, \qquad E_- = a_1 a_2, \qquad E_3 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \\ &F_+ = a_1 a_2^\dagger, \qquad F_- = a_1^\dagger a_2, \qquad F_3 = \frac{1}{2}(a_2^\dagger a_2 - a_1^\dagger a_1), \\ &U_+ = \frac{1}{2}{a_1^\dagger}^2, \qquad U_- = \frac{1}{2}{a_1}^2, \qquad V_+ = \frac{1}{2}{a_2^\dagger}^2, \qquad V_- = \frac{1}{2}{a_2}^2. \end{aligned} \tag{35}$$

Equation (13) can also be realized as

$$\mathcal{H}' = \frac{1-\mu}{2}a_1^\dagger a_1 + \frac{1+\mu}{2}a_2^\dagger a_2 + \rho(a_1 a_2 + a_1^\dagger a_2^\dagger) + \upsilon(a_1 a_2^\dagger + a_1^\dagger a_2). \tag{36}$$

Hereafter we leave out the index **k** for all the parameters and operators for convenience. So, for every **k** modes, utilizing the realization $a_j = (m\omega_j x_j + i p_j)/\sqrt{2m\omega_j}$, we can map a Hamiltonian of two-mode coupled harmonic oscillators corresponding hamiltonian $\mathcal{H}$ of Equation (11),

$$\mathcal{H}' = \sum_{j=1,2}\left[\frac{p_j^2}{2m} + \frac{\omega_j^2}{2}m x_j^2\right] + \lambda_1\omega_1\omega_2 m x_1 x_2 + \lambda_2\frac{p_1 p_2}{m}, \tag{37}$$

with

$$\frac{\sqrt{\omega_1\omega_2}}{2}(\lambda_1 - \lambda_2) = \rho, \quad \frac{\sqrt{\omega_1\omega_2}}{2}(\lambda_1 + \lambda_2) = \upsilon, \quad 1 - 2\omega_1 = 2\omega_2 - 1 = \mu. \tag{38}$$

Hamiltonian $\mathcal{H}$ and $\mathcal{H}'$ should have the same energy spectrum, i.e., in low excitation case, every **k** modes for XYZ antiferromagnetics in external magnetic field can be mapped onto a two-dimension coupled harmonic oscillators with $x$-$x$ and $p$-$p$ coupling, with the corresponding relation of the parameters Equation (38). One can see that, the system parameters $\upsilon$ and $\rho$ are linked to the coefficients of $x$-$x$ and $p$-$p$ coupling; while the role of the parameters $\mu$ (external magnetic field term) is to lift the energy degeneracy of the corresponding harmonic oscillators.

## 6. CONCLUDING REMARKS

In this paper, based on the introduction of an algebraic diagonalization method, the eigenstates of XYZ antiferromagnetics are revealed to be the number squeezed states in the framework of linear spin-wave approximation. Different from the traditional Bogoliubov-valatin transformation, we study the problem using the algebraic diagonalization method. The characteristic of this method lies in that, firstly, one can do calculation in virtue of the commutation relations of Lie algebra instead of the Heisenberg algebra; Secondly, the energy eigenstates are revealed to be squeezed number states, which had been studied in quantum optics field (Kim, 1989; Nieto, 1997; Yuen, 1976), making it convenient to study the physical properties of the system. It is shown that, the presence of magnetic field results in the non-degeneracy of the two excited magnon; even in the squeezed vacuum state, the spin direction on a sub-lattice is not necessarily the same (this disordering in a certain extent is attributed to the zero-point motion of the spin oscillator). Moreover, in the squeezed vacuum state, the influence of the external magnetic field upon the agree of average spin reversal of the two sub-lattice is absolutely the same. Some statistical property of the squeezed vacuum state, the second-order correlation functions, Mandel $Q$ parameters, violation of CIS are also discussed. By virtue of the algebra method, we illuminate that, XYZ antiferromagnetics under an external magnetic field can be mapped onto a two-dimension coupled harmonic oscillators with $x$-$x$ and $p$-$p$ coupling. Thus, the properties of the two-dimension coupled harmonic oscillators with $x$-$x$ and $p$-$p$ coupling, can readily be connected to the XYZ antiferromagnetics under an external magnetic field through adjusting parameter. Besides, our method can also be used to treat the sub-ferromagnetic case, and the case of existing anisotropic crystal magnetic field. The application of our results in related physical fields, as well as the realization and employment of squeezed number states in other physical systems, leave for further studies.

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